Fourier Analysis of the 2D Screened Poisson Equation for Gradient Domain Problems

Pravin Bhat[1] Brian Curless[1] Michael Cohen[1,2] C. Lawrence Zitnick[2]

[1]University of Washington [2]Microsoft Research

Abstract

We analyze the problem of reconstructing a 2D function that approximates a set of desired gradients and a data term. The combined data and gradient terms enable operations like modifying the gradients of an image while staying close to the original image. Starting with a variational formulation, we arrive at the "screened Poisson equation" known in physics. Analysis of this equation in the Fourier domain leads to a direct, exact, and efficient solution to the problem. Results using a DCT-based screened Poisson solver are demonstrated on several applications including painterly rendering, image re-lighting, image sharpening, and de-blocking of compressed images.

Gradient Domain Problem

Inputs


u


u_x


u_y

Application specific filtering

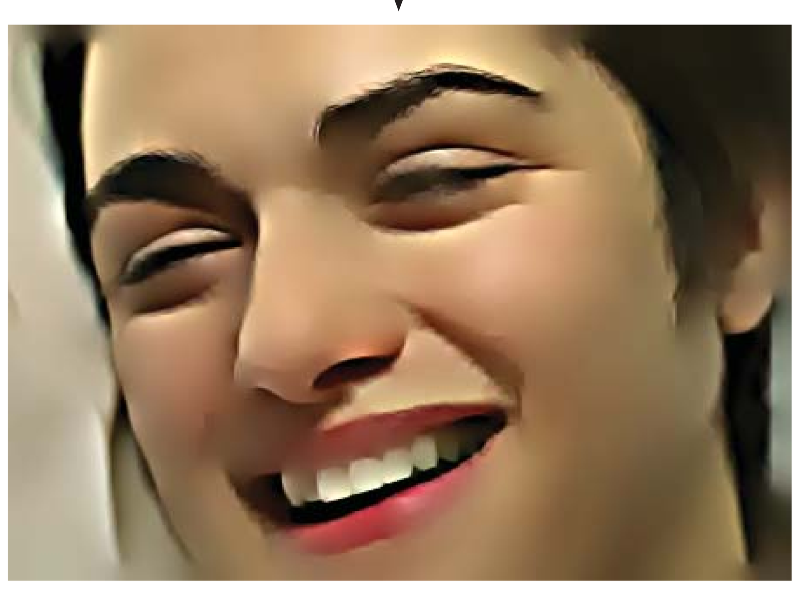
Constraints


d


g^x


g^y

Least squares solver


Solution - f

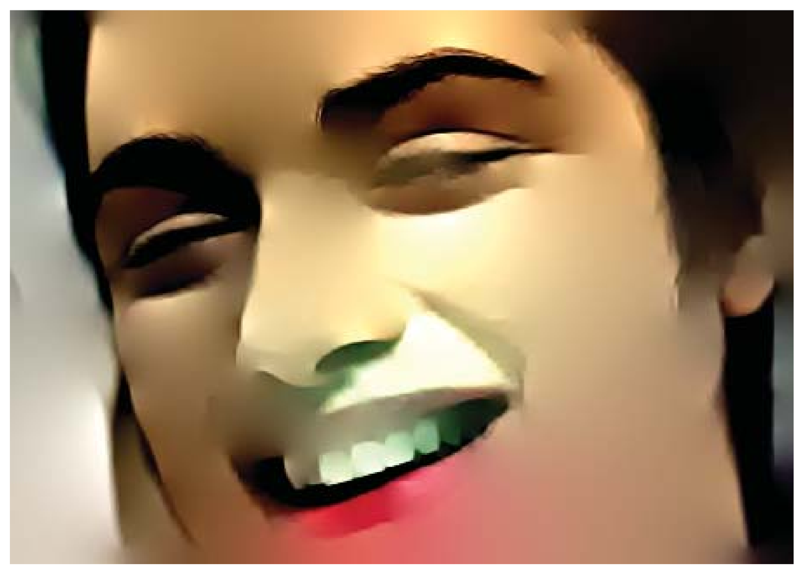
Fourier Domain Solutions

Poisson solution (previous work)

$$\arg\min_f \iint \|\nabla f - \mathbf{g}\|^2 \; dx \; dy$$

$$\nabla^2 f = \nabla\cdot\mathbf{g}$$ Poisson equation

$$F = \frac{-i2\pi s_x G^x - i2\pi s_y G^y}{4\pi^2(s_x^2 + s_y^2)}$$ Fourier domain solution


Result - f

Screened Poisson solution

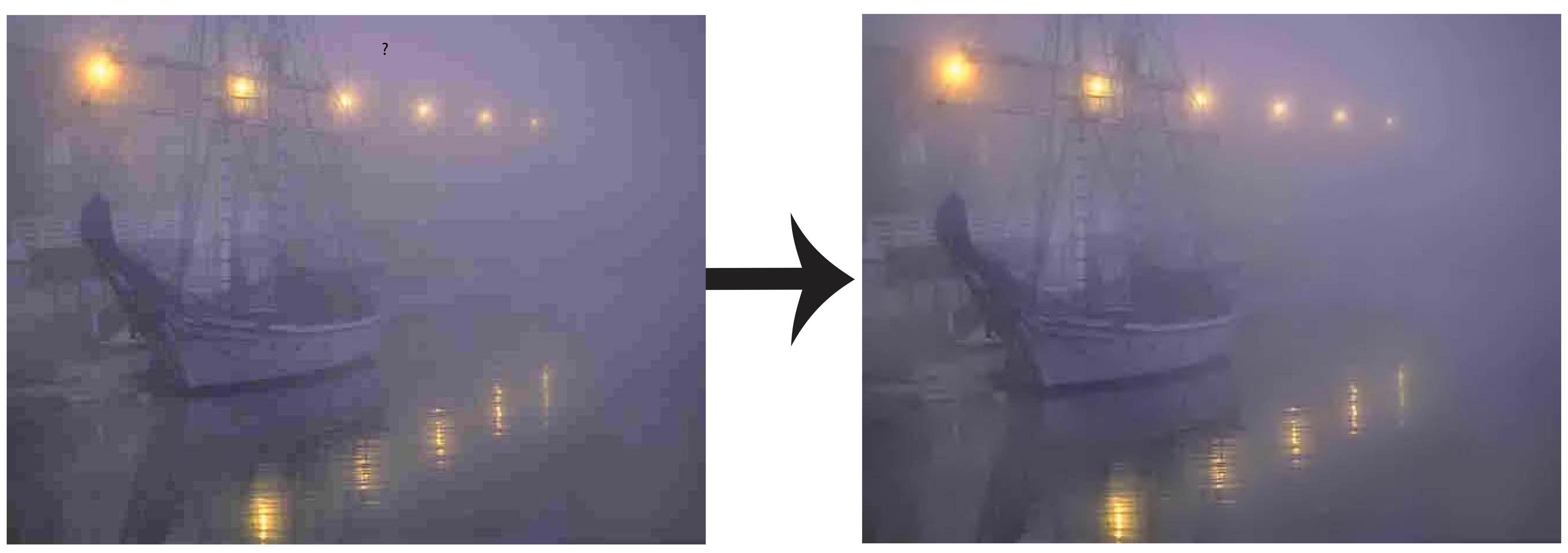
$$\arg\min_f \iint \lambda_d (f - d)^2 + \|\nabla f - \mathbf{g}\|^2 \; dx \; dy$$

$$\lambda_d f - \nabla^2 f = \lambda_d d - \nabla\cdot\mathbf{g}$$ Screened Poisson equation

$$F = \frac{\lambda_d D - i2\pi s_x G^x - i2\pi s_y G^y}{\lambda_d + 4\pi^2(s_x^2 + s_y^2)}$$ Fourier domain solution


Result - f

Applications


De-blocking

Saliency sharpening

Summary of Results (ask for a guided tour)

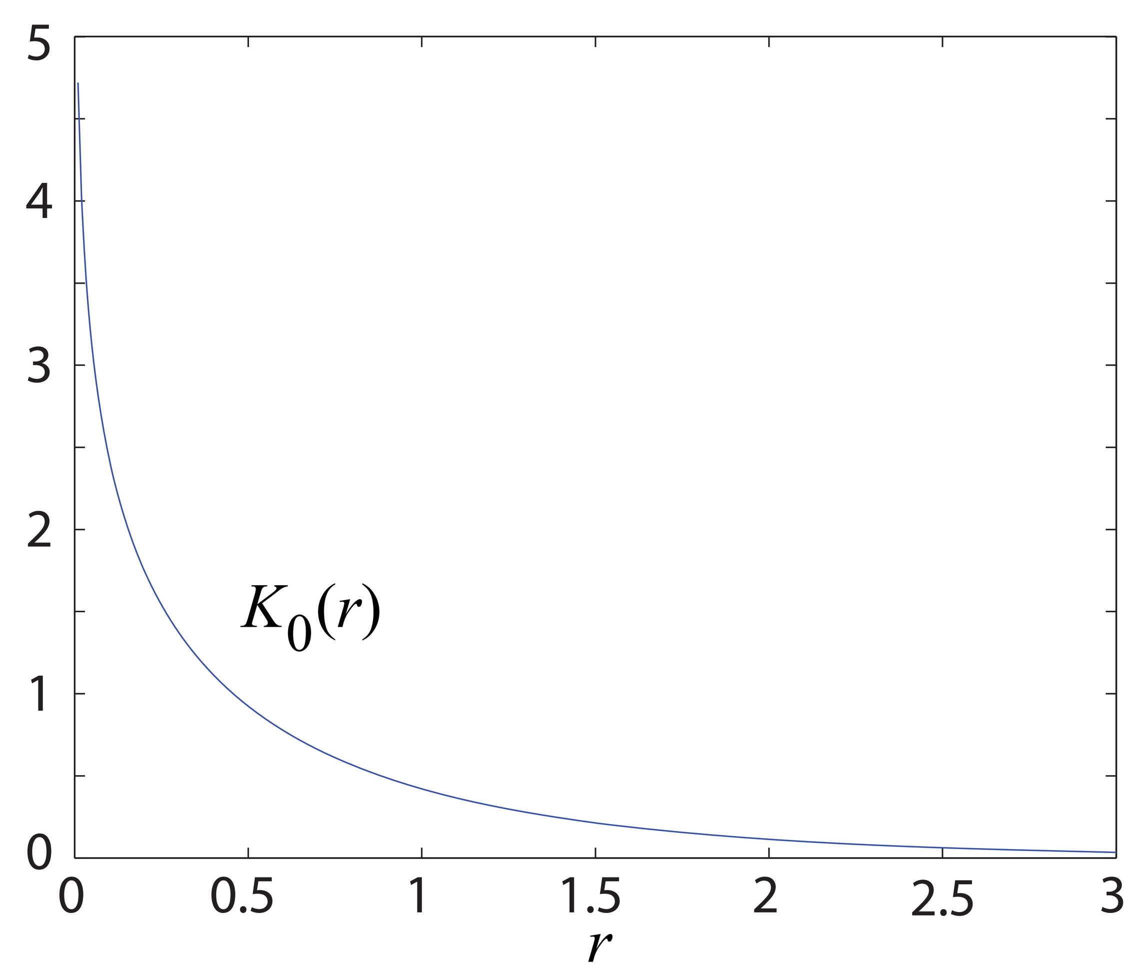
	General filtering through Screened Poisson equation	Sharpen filter through gradient amplification	Sharpen filter through Laplacian subtraction
Variational form	$\iint \lambda_d(f-d)^2 + \|\nabla f - \mathbf{g}\|^2 \; dx\; dy$	$\iint \lambda_d(f-u)^2 + \|\nabla f - c_s \nabla u\|^2 \; dx\; dy$	$\iint (f-u)^2 - 2\lambda_s \nabla f \cdot \nabla u \; dx\; dy$
Euler-Lagrange form	$\lambda_d f - \nabla^2 f = \lambda_d d - \nabla \cdot \mathbf{g}$	$\lambda_d f - \nabla^2 f = \lambda_d u - c_s \nabla^2 u$	$f = u - \lambda_s \nabla^2 u$
Fourier solution	$F = \dfrac{\lambda_d D - i2\pi s_x G^x - i2\pi s_y G^y}{\lambda_d + 4\pi^2(s_x^2 + s_y^2)}$	$F = \dfrac{1 + 4\pi^2(c_s/\lambda_d)(s_x^2 + s_y^2)}{1 + 4\pi^2(1/\lambda_d)(s_x^2 + s_y^2)} U$	$F = [1 + 4\pi^2 \lambda_s (s_x^2 + s_y^2)]\; U$
Spatial filter solution	$f(x,y) = \frac{1}{2\pi} K_0(2\pi\sqrt{\lambda_d(x^2+y^2)}) * (\lambda_d d - g_x^x - g_y^y)$	$f(x,y) = c_s u + \frac{\lambda_d(1-c_s)}{2\pi} K_0(2\pi\sqrt{\lambda_d(x^2+y^2)}) * u$	$f = u - \lambda_s \nabla^2 u$

sharpen filters

Screened Poisson: spatial filter solution

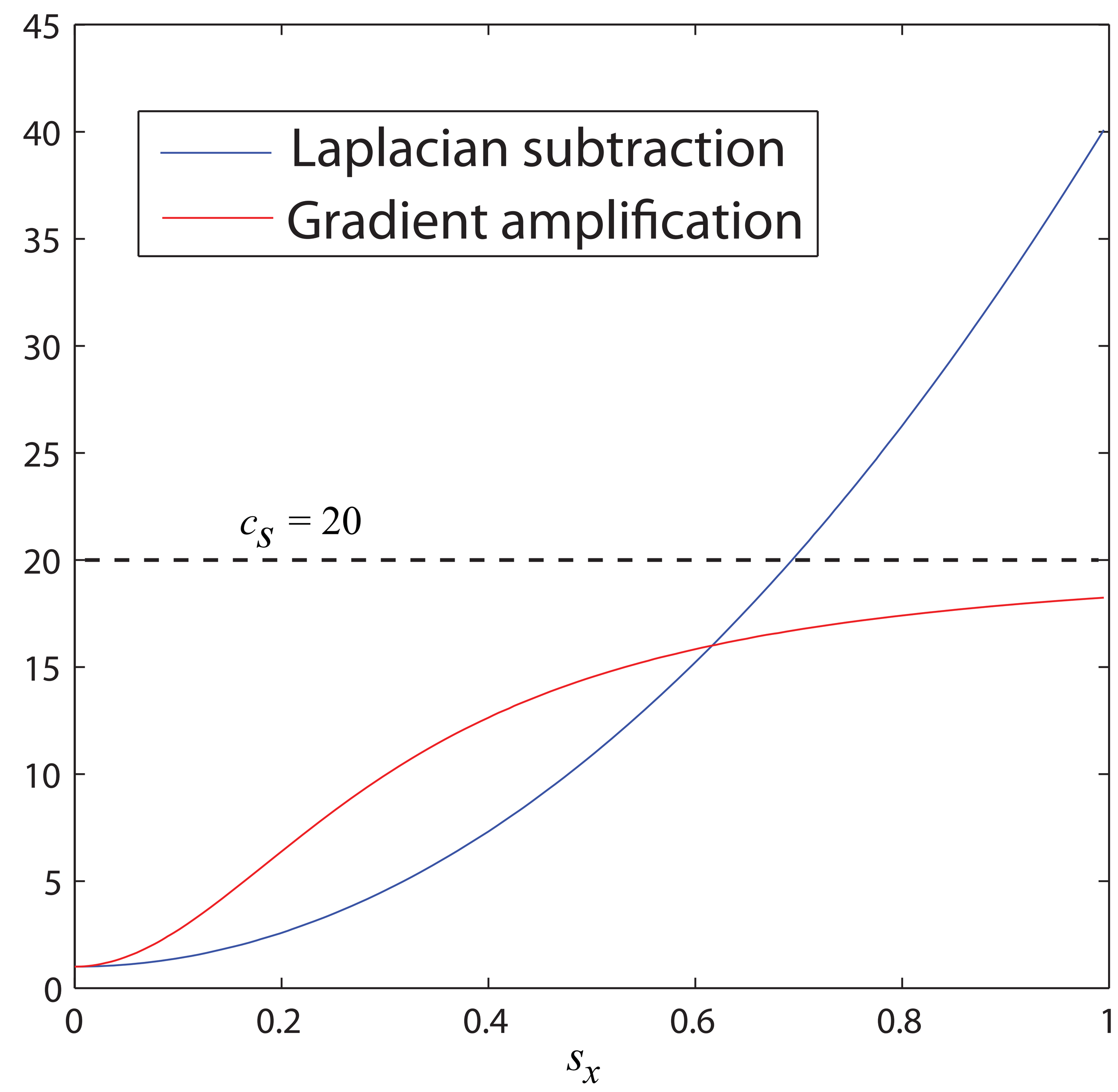
The screened Poisson equation can be solved via linear filtering: (1) scale the data image, (2) subtract the divergence of the supplied (generally non-integrable) gradient field, then (3) blur the result.

The blur function, shown below, is the zeroth order modified Bessel function of the second kind, $K_0(r)$. It is rotationally symmetric, and, while infinite at $r = 0$, is integrable.



Sharpen filter comparison: frequency domain

Here we plot sample frequency domain spectra for Laplacian subtraction and gradient amplification. For this illustration, we set the parameters as follows: $\lambda_s = 1$, $\lambda_d = 4$, and $c_s = 20$.



Performance evaluation on panoramic stitching

A comparison of memory and run-time performance of our method and other state-of-the-art methods for panoramic stitching (gradients only):

	Size	Time (s)					Memory (MB)				
Dataset	(MP)	FS	QT	SM	HB	LHAB	FS	QT	SM	HB	LHAB
St. Emilion	10	31	9	NA	3639	160	40	24	NA	362	1044
Beynac	12	22	8	17	3357	177	48	16	190	435	1252
Rainier	23	79	14	33	6446	268	92	27	110	620	1790
Sedona	36	85	29	NA	NA	NA	144	52	NA	NA	NA
Edinburgh	51	187	122	79	NA	NA	204	123	203	NA	NA
Crag	68	172	78	NA	NA	NA	272	96	NA	NA	NA
RedRock	88	270	118	118	NA	NA	352	112	133	NA	NA

* FS (our Fourier based solution using FFTW's DCT implementation)
* QT (Quadtree Compositing by *Agarwala et al. in SIGGRAPH 2007*)
* Out-of-core SM (Streaming Multigrid by *Kazhdan et al. in SIGGRAPH 2008*),
* HB (Heirarchical Basis Preconditioning by *Szeliski et al. in PAMI 1990*), and
* LHAB (Locally Adapted Heirarchical Function Preconditioning by *Szeliski et al. in SIGGRAPH 2006*).

Sharpen filter comparison: variational form

Laplacian subtraction - sharpen filter

$$\arg\min_f \iint (f-u)^2 - 2\lambda_s \nabla f \cdot \nabla u \; dx\; dy$$

Gradient amplification - sharpen filter

$$\arg\min_f \iint \lambda_d(f-u)^2 + \|\nabla f - c_s \nabla u\|^2 \; dx\; dy$$

$$\lambda_s = (c_s / \lambda_d)$$

$$\arg\min_f \iint (f-u)^2 - 2\lambda_s \nabla f \cdot \nabla u + \frac{1}{\lambda_d}\|\nabla f\|^2 \; dx\; dy$$